2
Mathematics of Space and Surface Grid Generation

2.1 Introduction
The purpose of this chapter is to provide a comprehensive mathematical background for the development of a set of differential equations that are geometry-oriented and are generally applicable for obtaining curvilinear coordinates or grids in intrinsically curved surfaces. To achieve this aim it is imperative to consider some geometrical results on curvilinear coordinates in the embedding space. The geometrical results are usually a consequence of some differential operations in the embedding space which also lead toward the theory of curves. The embedding space for non-relativistic problems is Euclidean or flat. Sections 2.2, 2.3, and 2.4 contain some basic results that are more fully explained in the books by Struijk [1], Kreyszig [2], Willmore [3], Eisenhart [4], Aris [5], and McConnell [6] among others, and in a monograph by Warsi [7]. In the course of development of the subject in this chapter, some elementary tools and results of tensor analysis have helped to provide concise results.
This chapter mainly focuses on one aspect of grid generation, which is the method of elliptic partial differential equations. It has been shown that the developed equations automatically satisfy some important results of the theory of surfaces. From this we conclude that the developed equations should be preferred to any other arbitrarily chosen set of equations to generate coordinates or grids in a surface. Another important outcome of these model equations is that the “fundamental theorem of surface theory” can be re-stated in a computationally realizable form. In other words, the proposed model equations can also be used to generate a surface if appropriate metric data** has been specified. Thus the proposed model equations have dual use, viz., generating the coordinate lines in a given surface, or generating a surface based on the metric data. Further, because of the elliptic nature of the equations, the generated grid lines will be smooth.

The idea of coordinate generation by using the elliptic partial differential equations in a plane is essentially due to Winslow [8]. However, if one stretches backward from Winslow to trace the foundations of the theory of coordinate generation by elliptic partial differential equations, then it is not possible to escape from the conclusion that the seed work was done by Allen [9], though in a different context. Later Chu [10] and Thompson, et al. [11] have used Winslow’s model for applications. In [11] extensive work was done to choose the coordinate control functions for application to a variety of problems. The application of the methods developed in [11] to extremely difficult problems involving geometries encountered in aeronautical engineering made the method of grid generation an important tool in CFD. Many years of work by a number of researchers and workers was published in a book [12]. Other books have followed in recent times ([13, 14]).

In an attempt to generalize the Winslow model of numerical coordinate generation, and further, to provide a mathematical foundation to the model equations, Warsi [15–18] has used the formulae of Gauss to arrive at the model equations as discussed in the cited references and in this article. These model equations are applicable for coordinate generation on generally curved surfaces with the coordinate generators (the control functions) appearing in them in a natural way. As noted earlier the same equations can also be used to generate a surface. For a plane these model equations reduce to those given in [8–11]. Some authors have also developed the surface coordinate generation model by using variational methods [19–21].

### 2.2 A Résumé of Differential Operations in Curvilinear Coordinates

For a presentation of a connected account of the theory of numerical coordinate mapping, it is imperative to review some basic concepts and formulae pertaining to the differential operations in curvilinear coordinates. As noted in the introduction, the formulae obtained by using simple tensor operations expose themselves effectively and in their full generality. Thus we use the symbol $x_i$, $i = 1, 2, \ldots, n$ to represent a curvilinear coordinate system in either a Euclidean or non-Euclidean $n$-space. In a Euclidean 3-space, denoted by $E^3$, one can introduce a rectangular Cartesian coordinate system $x_k$, $k = 1, 2, 3$, or $x_1 = x, x_2 = y, x_3 = z$, and the corresponding unit vectors $i_k, k = 1, 2, 3$, or $\hat{i}_1 = \hat{i}, \hat{i}_2 = \hat{j}, \hat{i}_3 = \hat{k}$. The position vector $\mathbf{r}$ is

$$
\mathbf{r} = \sum_{i=1}^{3} x_i \hat{i}_i
$$

(2.1a)

$$
\mathbf{r} = i x + j y + k z
$$

(2.1b)

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*For those readers who have not used tensor calculus in their works, the material presented here is, nevertheless, useful if the tensor quantities are viewed as abbreviations. For example, a Christoffel symbol is nothing but an abbreviated name of an algebraic sum of the first partial derivatives of the metric coefficients.

**Metric data means the first and second fundamental coefficients. Refer to Section 2.4.
(In general the repeated indices, when one is a subscript and the other a superscript, will imply summation over the range of index values. Exceptions to this rule will sometimes occur when the background system is rectangular Cartesian, as in Eq. 2.1a where both repeated indices are subscripts.) By introducing a general coordinate system \( x^i, i = 1, 2, 3 \) in \( E^3 \) and assuming the functions

\[
x_i = f_i(x^1, x^2, x^3), \quad i = 1, 2, 3
\]

(2.2a)

to be continuously differentiable and which are also invertible, i.e.,

\[
x^i = \Phi^i(x_1, x_2, x_3), \quad i = 1, 2, 3
\]

(2.2b)

we form the covariant base vectors

\[
a_i = \frac{\partial r}{\partial x^j}, \quad j = 1, 2, 3
\]

(2.3a)

where \( a_j \) is tangent to the coordinate curve \( x^j \). A system of reciprocal base vectors \( a^i \) are formed that satisfy the equations

\[
a^i \cdot a_j = \delta^i_j
\]

(2.3b)

where

\[
\delta^i_j = 0 \quad \text{if } i \neq j
\]

\[
= 1 \quad \text{if } i = j
\]

is the Kronecker symbol. (In a purely rectangular Cartesian setting it is a common practice to use \( \delta_j \) as the Kronecker symbol.) Since the coordinates \( x^i \) are independent among themselves, the simple result

\[
\frac{\partial x^i}{\partial x^j} = \delta^i_j
\]

leads one to the formula

\[
a^i = \text{grad} \ x^i
\]

(2.3c)

where

\[
\text{grad} = \nabla = \frac{\partial (\quad)}{\partial x^m} \quad i
\]

(2.4)

is the gradient operator.
2.2.1 Representations in Terms of $a_i$ and $a^i$

All quantities that follow certain transformation of coordinate rules are called tensors. Tensors of various orders (ranks) can either be formed or appear naturally. In particular, scalars and vectors are tensors of order zero and one, respectively. A vector $u$ can be represented in either of the following forms:

\[ u = u^i a_i \]  \hspace{1cm} (2.5a)  \\
\[ = u_i a^i \]  \hspace{1cm} (2.5b)  \\
\[ U i + V j + W k \]  \hspace{1cm} (2.5c)

In Eqs. 2.5a, 2.5b $u^i$ and $u_i$ are the contravariant and covariant components of $u$, respectively. In the same fashion a tensor $T$ of second order can be represented in any one of the following forms:

\[ T = T^{ij} a_i a^j \]  \hspace{1cm} (2.6a)  \\
\[ = T_{ij} a^i a^j \]  \hspace{1cm} (2.6b)  \\
\[ = T_{ij} a^i a^j \]  \hspace{1cm} (2.6c)  \\
\[ = T_{ij} a_i a^j \]  \hspace{1cm} (2.6d)

Here $T^{ij}$ are the contravariant components and $T_{ij}$ are the covariant components of $T$. In Eqs. 2.6c, 2.6d the components are of the mixed type. Further $a_i a^j$ is the dyadic product of the vectors $a_i$ and $a^j$. A unit tensor $I$ has units on the main diagonal and zeros elsewhere. Thus using either Eq. 2.6c or Eq. 2.6d we have

\[ \tilde{I} = \delta^j_i a^i a^j = \delta^i_j a^i a^j \]

In short,

\[ \tilde{I} = a^i a^j = a^i a_i \]  \hspace{1cm} (2.7)

The transpose of the tensor $T$ is denoted as $T^T$, and has the representation

\[ \tilde{T}^T = T^{ij} a_i a^j = T^{ij} a^j a_i \]  \hspace{1cm} (2.8)

and similarly with the other representations. A tensor is symmetric if

\[ \tilde{T}^T = T \]  \hspace{1cm} (2.9a)
and skew-symmetric if

\[ \overline{T}^T = -\overline{T} \]  

(2.9b)

Vectors and tensors in the rectangular Cartesian system can be written in a straightforward manner using summation on repeated subscripts, e.g., [22].

### 2.2.2 Differential Operations

Let the position vector \( \vec{r} \) be expressed in terms of the curvilinear coordinates \( x^i \). The first differential \( d\vec{r} \) is then

\[ d\vec{r} = \frac{\partial \vec{r}}{\partial x^i} dx^i \]

Using Eq. 2.3a,

\[ d\vec{r} = a_i dx^i \]  

(2.10)

On comparison with Eq. 2.5a we note that \( dx^i \) are the contravariant components of the differential displacement vector \( d\vec{r} \). It must, however, be noted that \( x^i \) are not the contravariant components of any vector.

Let \( \phi(x^1, x^2, x^3) \) be a scalar point function. Then its first differential is

\[ d\phi = \frac{\partial \phi}{\partial x^i} dx^i \]  

(2.11a)

From Eq. 2.10, using Eq. 2.3b we have

\[ dx^i = a^i \cdot d\vec{r} \]  

(2.11b)

which when used in Eq. 2.11a yields

\[ d\phi = (\nabla \phi) \cdot d\vec{r} \]  

(2.11c)

where

\[ \nabla \phi = \frac{\partial \phi}{\partial x^i} a^i \]  

(2.11d)

is the gradient of \( \phi \), and is a vector.

Let \( \vec{u} \) be a vector function of position; then its first differential is

\[ d\vec{u} = \frac{\partial \vec{u}}{\partial x^i} dx^i \]
Using Eq. 2.11b, we have

\[ d\mathbf{u} = \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}^i} \mathbf{a}^i \right) \cdot d\mathbf{r} \]

We shall use the definition of the gradient of a vector as

\[ \text{grad } \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}^i} \mathbf{a}^i \]  \hspace{1cm} (2.12)

so that

\[ d\mathbf{u} = \left( \text{grad } \mathbf{u} \right) \cdot d\mathbf{r} \]  \hspace{1cm} (2.13)

The divergence of a vector field \( \mathbf{u} \) is obtained by adding the diagonal terms of the tensor grad \( \mathbf{u} \), which in vector operational form is

\[ \text{div } \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}^i} \cdot \mathbf{a}^i \]  \hspace{1cm} (2.14)

Taking a lead from Eq. 2.14, the divergence of a tensor is

\[ \text{div } \mathbf{T} = \frac{\partial \mathbf{T}}{\partial \mathbf{x}^i} \cdot \mathbf{a}^i \]  \hspace{1cm} (2.15)

To complete this discussion, the curl of a vector field \( \mathbf{u} \) is defined as

\[ \text{curl } \mathbf{u} = \mathbf{a}^i \times \frac{\partial \mathbf{u}}{\partial \mathbf{x}^i} \]

### 2.2.3 Metric Tensor and the Line Element

In \( E^3 \) we introduce a system of curvilinear coordinates \( x^i \). The differential displacement vector is then given by Eq. 2.10 and the length element \( ds \) is given by

\[ ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \left( \mathbf{a} \cdot \mathbf{a} \right) dx^i dx^j \]

Writing

\[ g_{ij} = \mathbf{a} \cdot \mathbf{a} \]  \hspace{1cm} (2.16)

we obtain

\[ ds^2 = g_{ij} dx^i dx^j \]  \hspace{1cm} (2.17)
The coefficient \( g_{ij} \) are the covariant components of the metric tensor. Though Eq. 2.17 has been obtained for a Euclidean space, it is applicable to both the Euclidean and non-Euclidean spaces. In fact, Eq. 2.17 forms the one and the only postulate of Riemannian geometry. Obviously, \( g_{ij} \) are symmetric components, i.e.,

\[
g_{ij} = g_{ji}
\]

and the determinant of the matrix formed by \( g_{ij} \) is

\[
g = \det(g_{ij})
\]

which is strictly positive for \( E^3 \). The contravariant components of the metric tensor are

\[
g^{ij} = a^i \cdot a^j
\]

which are easily obtained in terms of \( g_{ij} \) as

\[
g^{ij} = \left( g_{rp}g_{lt} - g_{rt}g_{lp}\right) / g
\]

where the groups \((i, r, l)\) and \((j, p, t)\) separately assume values in the cyclic permutations of 1, 2, 3, in this order. Introducing the following subdeterminants,

\[
G_1 = g_{22}g_{33} - (g_{23})^2 \\
G_2 = g_{11}g_{33} - (g_{13})^2 \\
G_3 = g_{11}g_{22} - (g_{12})^2 \\
G_4 = g_{13}g_{23} - g_{12}g_{33} \\
G_5 = g_{12}g_{23} - g_{13}g_{22} \\
G_6 = g_{12}g_{13} - g_{11}g_{23}
\]

we have on using Eq. 2.20:

\[
g^{11} = G_1 / g, \quad g^{22} = G_2 / g, \quad g^{33} = G_3 / g \\
g^{12} = G_4 / g, \quad g^{13} = G_5 / g, \quad g^{23} = G_6 / g
\]

As is shown in the cited references, e.g. [2, 7],

\[
g_{ji}g^{kl} = \delta^k_j
\]

\[
a^i = g_{ik} a^k
\]

\[
a^l = g^{lk} a_k
\]
Writing \( x^1 = \xi, x^2 = \eta, x^3 = \zeta \), and denoting a partial derivative by a variable subscript, one of the expanded forms of \( \sqrt{g} \) is

\[
\left( y_\xi z_\eta - y_\eta z_\xi \right) x_\zeta + \left( x_\eta z_\xi - x_\xi z_\eta \right) y_\zeta + \left( x_\xi y_\eta - x_\eta y_\xi \right) z_\zeta = \sqrt{g}
\]  

(2.25b)

Using Eq. 2.21, we also have

\[
g = G_1 g_{11} + G_4 g_{12} + G_5 g_{13} = G_2 g_{22} + G_4 g_{12} + G_6 g_{23} = G_3 g_{33} + G_5 g_{13} + G_6 g_{23}
\]  

(2.25c)

Other representations of the base vectors are

\[
a^i \frac{\sqrt{g}}{2} \left( a^j \times a^k \right) e^{ijk}
\]  

(2.26a)

\[
a^i \frac{\sqrt{g}}{2} \left( a^i \times a^k \right) e_{ijk}
\]  

(2.26b)

where \( e^{ijk} \) and \( e_{ijk} \) are the permutation symbols. In terms of the metric tensor, the unit tensor defined in Eq. 2.7 is

\[
\tilde{I} = g_{ij} a^i a^j
\]  

(2.27a)

\[
= g^{ij} a_i a_j
\]  

(2.27b)

\[
= \delta^j_i a^j
\]  

(2.27c)

Using Eq. 2.24 in Eq. 2.5, we have

\[
u^i = g^{ik} u_k
\]  

(2.28a)

and

\[
u_i = g_{ik} u^k
\]  

(2.28b)

### 2.2.4 Differentiation of the Base Vectors

The main aim is to express the partial derivatives of the base vectors in terms of the base vectors. First, from the definition of the covariant base vectors, Eq. 2.3a, it is readily obvious that

\[
\frac{\partial a^i}{\partial x^j} = \frac{\partial a^i}{\partial x^j}
\]  

(2.29)
Using this result and the simple derivations given in [7] we have the following results:

\[
\frac{\partial a_i}{\partial x^j} = [ij, k]a^k
\]  

(2.30a)

\[
= \Gamma^k_{ij}a_k
\]  

(2.30b)

where the abbreviations

\[
[ij, k] = \frac{1}{2} \left( \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right)
\]  

(2.31a)

and

\[
\Gamma^k_{ij} = g^{ik}[ij, s]
\]  

(2.31b)

are called the Christoffel symbols of the first and second kind, respectively. Note that \([ij, k] = [ji, k]\) and \(\Gamma^k_{ij} = \Gamma^k_{ji}\). Eq. 2.30b can also be stated as

\[
\frac{\partial^2 r}{\partial x^i \partial x^j} = \Gamma^k_{ij}a_k
\]  

(2.32)

To obtain the partial derivatives of the contravariant base vectors \(a^i\), we differentiate Eq. 2.3b with respect to any coordinate, say \(x^i\), and use the previous results to obtain

\[
\frac{\partial a^i}{\partial x^k} = -\Gamma^i_{jk}a^j
\]  

(2.33)

Taking the dot product of Eq. 2.33 with \(a^k\) and using the definition in Eq. 2.3c, we readily get

\[
\nabla^2 x^i = -g^{jk}\Gamma^i_{jk}
\]  

(2.34)

where

\[
\nabla^2 = \frac{\partial^2}{\partial x^i \partial x^i}
\]

is the Laplacian operator and \(x_m\) are the Cartesian coordinates.

### 2.2.5 Covariant and Intrinsic Derivatives

When one takes the partial derivative of a vector in its entity form, i.e.,

\[
\frac{\partial u}{\partial x^k} = \frac{\partial}{\partial x^k} \left( u^i a_i \right)
\]
and uses Eq. 2.30b, the result is

\[ \frac{\partial u}{\partial x^k} = u_i^k a_i \]  \hspace{1cm} (2.35a)

where

\[ u_i^k = \frac{\partial u_i^j}{\partial x^k} + \Gamma_k^j u^j \]  \hspace{1cm} (2.35b)

is called the covariant derivative of a contravariant component. A semicolon before an index implies covariant differentiation. Similarly,

\[ \frac{\partial u}{\partial x^k} = \frac{\partial}{\partial x^k} \left( u_i a^i \right) \]

and then on using Eq. 2.33, one gets

\[ \frac{\partial u}{\partial x^k} = u_{i;k} a^i \]  \hspace{1cm} (2.36a)

where

\[ u_{i;k} = \frac{\partial u_i}{\partial u^k} - \Gamma_k^j u^j \]  \hspace{1cm} (2.36b)

is called the covariant derivative of a covariant component. The idea of covariant differentiation can be extended to tensors of any order. Refer to [5] and [22] for some explicit formulae for a second-order tensor. In particular it can be shown that the covariant derivatives of the metric tensor components are zero. That is

\[ g_{ij;k} = 0, \ g^{'ij}_{;k} = 0 \]

These two equations yield explicit formulae for the partial derivatives of the covariant and contravariant metric components, which are

\[ \frac{\partial g_{ij}}{\partial x^k} = \Gamma^r_{ik} g_{rj} + \Gamma^r_{jk} g_{ri} \]  \hspace{1cm} (2.37a)

and

\[ \frac{\partial g^{'ij}}{\partial x^k} = -\Gamma^r_{rk} g^{'ij} - \Gamma^r_{jk} g^{'ri} \]  \hspace{1cm} (2.37b)

Let \( G^m \) be the cofactor of \( g_{rm} \) in the determinant \( g \). Then

\[ g^{'r} = g_{pm} G^m \]
and

\[ G^{m} = g g^{m} \]

Thus

\[ \frac{\partial g}{\partial x^j} = g g^{m} \frac{\partial g}{\partial x^j} \]  \hspace{1cm} (2.38)

Using Eq. 2.37a in Eq. 2.38, one readily obtains

\[ \Gamma^r_{ij} = \frac{1}{2g} \frac{\partial g}{\partial x^j} \]  \hspace{1cm} (2.39a)

\[ = \frac{\partial}{\partial x^j} \left( \ln \sqrt{g} \right) \]  \hspace{1cm} (2.39b)

Using Eqs. 2.3b, 2.30b, and 2.39a in Eq. 2.14, the formula for the divergence of a vector \( \vec{u} \) becomes

\[ \text{div} \ u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left( \sqrt{g} u^j \right) \]  \hspace{1cm} (2.40)

Similarly, the formulae for the divergence of a tensor can be developed.

Let the curvilinear coordinates \( x^i \) be functions of a single parameter \( t \), i.e.,

\[ x^i = x^i(t), \quad t_0 \leq t \leq t_1 \]

Then \( \vec{u} \) becomes a function of \( t \), i.e.,

\[ \vec{u}(x^i) = u(x^i(t)) \]

and the total derivative of \( \vec{u} \) with \( t \) is

\[ \frac{d\vec{u}}{dt} = \frac{d}{dt} \left( \begin{array}{c} u^i \\alpha \\ -i \end{array} \right) \]

\[ = \frac{du^i}{dt} = \alpha + u^i \frac{d\alpha}{dt} \]

Using the chain rule of partial differentiation and the definition of the covariant derivative, one obtains

\[ \frac{d\vec{u}}{dt} = \left( \frac{\partial u^i}{\partial t} + u^i_j \frac{dx^j}{dt} \right) \alpha \]

The intrinsic derivative of \( u^i \) is defined as

\[ \frac{\delta u^i}{\delta t} = \frac{\partial u^i}{\partial t} + u^i_j \frac{dx^j}{dt} \]  \hspace{1cm} (2.41)
and then

\[
\frac{du}{dt} = \delta u^a_i \partial a\frac{\partial a}{\partial t} - i
\]  

(2.42)

### 2.2.6 Laplacian of a Scalar

Let \( \phi (x^1, x^2, x^3) \) be a scalar. The Laplacian of \( \phi \) is defined as

\[
\nabla^2 \phi = \text{div} (\text{grad} \phi)
\]  

(2.43)

From Eq. 2.11d, the components \( \frac{\partial \phi}{\partial x_i} \) are the covariant components of the vector \( \text{grad} \phi \). According to Eq. 2.28a, the contravariant components are

\[
g^{ij} \frac{\partial \phi}{\partial x^j}
\]

Thus using Eq. 2.40,

\[
\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial \phi}{\partial x^j} \right)
\]  

(2.44)

which is one of the form for the Laplacian. Another form can be obtained by opening the differentiation on right-hand side and using Eqs. 2.37b and 2.39a, or else using Eq. 2.11d in Eq. 2.14 and using the preceding developed formulae. In either case, we get

\[
\nabla^2 \phi = g^{ij} \left( \frac{\partial^2 \phi}{\partial x^i \partial x^j} - \Gamma^{k}_{ij} \frac{\partial \phi}{\partial x^k} \right)
\]  

(2.45a)

or, by using Eq. 2.34,

\[
\nabla^2 \phi = g^{ij} \frac{\partial^2 \phi}{\partial x^i \partial x^j} + \left( \nabla^2 x^k \right) \frac{\partial \phi}{\partial x^k}
\]  

(2.45b)

Note that if \( \phi = x^r \), a curvilinear coordinate, then from Eq. 2.45a,

\[
\nabla^2 x^r = -g^{ij} \Gamma^r_{ij}
\]

which is Eq. 2.34.

### 2.3 Theory of Curves

Practically all standard texts on differential geometry describe the theory of curves in formal details [1–6]. This section is intended to supplement the textual material in later sections for reference.

In \( E^3 \) using the rectangular Cartesian coordinates \( x_m, m = 1, 2, 3 \), the position vector at a point on the curve is stated as a function of an arbitrary parameter \( t \) as

\[
r(t) = x_m(t) i_m, \quad t_0 \leq t \leq t_1
\]
The main assumption here is that at least one derivative,

\[ \dot{x}_m = \frac{dx_m}{dt}, \quad m = 1, 2, 3 \]

is different from zero. A simple example of the parametric equation of a curve is that of a straight line, which is

\[ r(t) = a + b t \]

where \( a \) and \( b \) are constant vectors with the components of \( b \) being proportional to the direction cosines of the line.

On a curve the arc length from a point \( P_0 \) of parameter \( t_0 \) to a point \( P \) of parameter \( t \) can be obtained by using Eq. 2.17 in Cartesian coordinates. Thus,

\[ ds^2 = dr \cdot dr = \dot{r} \cdot \dot{r} (dt)^2 \]

so that

\[ s(t) = \int_{t_0}^{t} \sqrt{\dot{r} \cdot \dot{r}} dt \]

If instead of \( t \) one takes the arc length as a parameter, then from Eq. 2.46

\[ t \cdot t = 1 \]  \hspace{1cm} (2.47a)

where

\[ t = \frac{dr}{ds} \]  \hspace{1cm} (2.47b)

From Eq. 2.47a it is obvious that \( t(s) \) is a unit vector tangent to the curve. Further,

\[ \dot{r} = t \frac{ds}{dt} \]  \hspace{1cm} (2.47c)

is also a tangent vector. Differentiating Eq. 2.47a, we get

\[ t \cdot \frac{dt}{ds} = 0 \]

Writing

\[ \hat{k} = \frac{dt}{ds} \]  \hspace{1cm} (2.48)
we note that the vectors $\mathbf{t}$ and $\mathbf{k}$ are orthogonal. The vector $\mathbf{k}$ is the curvature vector because it expresses the rate of change of the unit tangent vector as one follows the curve. Now forming the unit vector

$$p = \frac{\mathbf{k}}{k}$$  \hspace{1cm} (2.49a)

where

$$k = \|\mathbf{k}\|$$  \hspace{1cm} (2.49b)

is the curvature of the curve at a point. The unit vector $p$ is called the principal normal vector. The plane containing $\mathbf{t}$ and $p$ is called the osculating plane.

Another vector $\mathbf{b}$ is now formed as

$$b = \mathbf{t} \times p$$  \hspace{1cm} (2.50)

The triad of vectors $\mathbf{t}$, $p$, $\mathbf{b}$, in this order, form a right-hand system of unit vectors at a point of the curve. Besides the osculating plane the two other planes, termed the normal plane and the rectifying plane, are shown in Figure 2.1.

The vector $\mathbf{b}$ is called the binormal vector and is associated with the torsion of the space curve. Based on simple arguments, e.g. [7], we can obtain the famous formulae of Frenet, or of Serret–Frenet, which are

$$\frac{dt}{ds} = kp$$  \hspace{1cm} (2.51a)

$$\frac{dp}{ds} = -kt + \tau b$$  \hspace{1cm} (2.51b)

$$\frac{db}{ds} = -\tau p$$  \hspace{1cm} (2.51c)
The scalar $\tau$ is called the torsion of a curve at a point and it is zero for plane curves.

Eqs. 2.51 are fundamental to the theory of curves. In fact, the fundamental theorem for space curves is stated as follows. “If $s > 0$ is the arc length along a curve and the functions $k(s)$ and $\tau(s)$ are single-valued and prescribed functions of $s$, then the solution of Eqs. 2.51 yields a space curve which is unique except for its position in space.” For prescribed $k(s)$ and $\tau(s)$ Eqs. 2.51 can be solved in analytical forms for some very small number of cases. Eqs. 2.51 form a set of nine scalar equations, and if the initial conditions at some $s = s_0$ are prescribed for $t$ and $p$ (initial condition for $b$ can then be obtained from Eq. 2.50), then according to the theory of existence of the ordinary differential equations, the set of nine equations can be solved by any standard numerical method, such as the Runge–Kutta method. If $k$ and $\tau$ are prescribed in terms of some other parameter $t$, then the same program can be slightly altered by prescribing $ds/dt$ and replacing $k(s)$ by $k(t)$, etc., in the program.

### 2.3.1 A Collection of Usable Formulae for Curves

The formulae of curvature and torsion in terms of the arc length $s$ for a curve $\gamma(s)$ are as follows:

\[ k(s) = \left( \frac{d^2 r}{ds^2} \cdot \frac{d^3 r}{ds^3} \right)^{1/2} \]  

\[ \tau(s) = \rho^2 \frac{dr}{ds} \left( \frac{d^2 r}{ds^2} \cdot \frac{d^3 r}{ds^3} \right) \]  

where

\[ \rho(s) = \frac{1}{k(s)} \]

is the radius of curvature. If the curve is expressed in terms of a parameter $t$ as $\gamma(t)$, then denoting differentiation with $t$ by a dot, we have

\[ k(t) = \left[ \left( \dot{r} \cdot \ddot{r} \right) \left( \dddot{r} \cdot \dddot{r} \right)^{1/2} \right] \right] \left( \dot{r} \cdot \dddot{r} \right)^{3/2} \]  

\[ \tau(t) = \rho^2 \dot{r} \cdot \left( \dddot{r} \times \dddot{r} \right) \left( \dot{r} \times \dddot{r} \right)^3 \]  

Let a space curve be defined as the intersection of the two surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$. Then the unit tangent vector of the curve is given by \[t\]

\[ t = \left( i J_1 + j J_2 + k J_3 \right) \left( J_1^2 + J_2^2 + J_3^2 \right)^{1/2} \]

where

\[ J_1 = f_y g_z - f_z g_y, \quad J_2 = f_z g_x - f_x g_z, \quad J_3 = f_x g_y - f_y g_x \]

and a variable subscript denotes a partial derivative.

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2.4 Geometrical Elements of the Surface Theory

The theory of surfaces embedded in \( E^3 \) was developed with all its essential aspects in the 19th century. Almost all of the useful concepts and formulae presently used in engineering and applied sciences were developed by Gauss, Monge, Darboux, Beltrami, and Christoffel, just to name a few. For a detailed discussion of the topics discussed in this section, the reader is referred to Refs. [1–3].

In the theory of surfaces embedded in \( E^3 \) we can either use the rectangular Cartesian coordinates \( x_m \) or some general coordinates \( x^i \). For the sake of generality, let us first use a general system of coordinates \( x^i \). A surface is then defined parametrically by the use of two parameters \( u^\alpha = (u^1, u^2) \) as

\[
x^i = x^i(u^1, u^2), \quad i = 1, 2, 3
\]  

(2.55a)

The functions \( x^i \) defined in Eq. 2.55a are continuously differentiable with respect to the parameters \( u^1 \) and \( u^2 \), and the matrix

\[
\left[ \frac{\partial x^i}{\partial u^\alpha} \right]
\]

is of rank two, i.e., at least one square subdeterminant is not zero. From Eq. 2.55a,

\[
dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha
\]  

(2.55b)

where the Greek indices assume values 1 and 2. Also, the displacement vector \( dr \), which belongs both to the surface and the embedding space \( E^3 \), can be represented either as

\[
dr = \frac{\partial r}{\partial x^i} dx^i = a_i dx^i
\]  

(2.55c)

or as

\[
dr = \frac{\partial r}{\partial u^\alpha} du^\alpha
\]  

(2.55d)

The element of length

\[
 ds^2 = dr \cdot dr
\]

from Eq. 2.17, or alternatively from Eq. 2.55c by using Eq. 2.55b, can be stated as

\[
 ds^2 = a_{\alpha\beta} du^\alpha du^\beta
\]  

(2.56)

where

\[
a_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}
\]  

(2.57)
Obviously $a_{\alpha\beta}$ are symmetric. Since the embedding space is Euclidian, one can also use the rectangular Cartesian coordinates $x_m$ in place of the curvilinear coordinates $x^i$. In such a case $g_{ij} = \delta_{ij}$ and from Eq. 2.57,

$$a_{\alpha\beta} = \frac{\partial x_m}{\partial u^\alpha} \frac{\partial x_m}{\partial u^\beta} = \frac{\partial r}{\partial u^\alpha} \frac{\partial r}{\partial u^\beta}$$

From here onward we shall return to the previous symbolism and use $g_{\alpha\beta}$ in place of $a_{\alpha\beta}$ so that

$$g_{\alpha\beta} = \frac{\partial r}{\partial u^\alpha} \cdot \frac{\partial r}{\partial u^\beta}$$ (2.58)

and Eq. 2.56 is written as

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta$$ (2.59)

which gives an elemental arc on a surface of parameters/coordinates $u^1, u^2$. The metric, Eq. 2.59, for an element of length in the surface is called the “first fundamental form.” For the purpose of having expanded formulae we write $x_1 = x, x_2 = y, x_3 = z; u^1 = \xi, u^2 = \eta$ and then from Eq. 2.58:

$$g_{11} = x^2 + y^2 + z^2$$ (2.60a)

$$g_{12} = x_{\xi} x_{\eta} + y_{\xi} y_{\eta} + z_{\xi} z_{\eta}$$ (2.60b)

$$g_{22} = x_{\eta}^2 + y_{\eta}^2 + z_{\eta}^2$$ (2.60c)

$$G_3 = g_{11} g_{22} - (g_{12})^2$$ (2.60d)

where a variable subscript implies a partial derivative. Further, similar to Eq. 2.23 we have

$$g_{\alpha\beta} g^{\alpha\gamma} = \delta^\gamma_\beta$$ (2.61a)

so that

$$g^{11} = g_{22} / G_3,\ g^{12} = g_{21} = -g_{12} / G_3,\ g^{22} = g_{11} / G_3$$ (2.61b)

The vectors

$$a_\alpha = \frac{\partial r}{\partial u^\alpha},\ \alpha = 1, 2$$ (2.62)

are the covariant surface base vectors and they form a tangent vector field. The angle $\theta$ between the coordinate lines $\xi = u^1$ and $\eta = u^2$ at a point in the surface is obviously given by

$$\cos \theta = \frac{a_1 \cdot a_2}{\| a_1 \| \| a_2 \|} = \frac{g_{12}}{\sqrt{g_{11} g_{22}}}$$ (2.63a)
and

\[ |a \times a|_{1-2}^2 = g_{11} g_{22} \sin^2 \theta \]
\[ = g_{11} g_{22} \left(1 - \cos^2 \theta\right) \]

Thus, using Eq. 2.63a, we have

\[ |a \times a|_{1-2}^2 = g_{11} g_{22} - (g_{12})^2 \]
\[ = G_3 \] (2.63b)

Coordinates in the surface at a point are orthogonal if \( g_{12} = 0 \) at that point.

The surface base vectors in Eq. 2.62 define the unit normal vector \( \mathbf{n} \) at each point of the surface through the equation

\[ n = \frac{1}{\sqrt{G_3}} \left( a \times a \right) \]

Thus

\[ n = \frac{1}{\sqrt{G_3}} \left( a \times a \right) \] (2.64)

The rectangular Cartesian components of \( \mathbf{n} \) denoted by \( X, Y, Z \) are

\[ X = J_1 / \sqrt{G_3}, \quad Y = J_2 / \sqrt{G_3}, \quad Z = J_3 / \sqrt{G_3} \] (2.65)

where

\[ J_1 = y_\xi z_\eta - y_\eta z_\xi, \quad J_2 = x_\eta z_\xi - x_\xi z_\eta, \quad J_3 = x_\xi y_\eta - x_\eta y_\xi \]

### 2.4.1 The Surface Christoffel Symbols

The surface Christoffel symbols can be formed by the same technique as noted in Section 2.2, independent of any other consideration. For clarity in the analysis to follow, we shall denote the surface Christoffel symbols of the second kind by \( \Upsilon^\sigma_{\alpha\beta} \). The formula is

\[ \Upsilon^\sigma_{\alpha\beta} = g^{\sigma\delta} [\alpha\beta, \delta] \] (2.66)

where

\[ [\alpha\beta, \delta] = \frac{1}{2} \left( \frac{\partial g_{\alpha\beta}}{\partial u^\delta} + \frac{\partial g_{\alpha\delta}}{\partial u^\beta} - \frac{\partial g_{\beta\delta}}{\partial u^\alpha} \right) \] (2.67)

and \([\alpha\beta, \delta]\) are the surface Christoffel symbols of the first kind. The technique mentioned above can concisely be stated as follows:
Obviously (similar to Eq. 2.29),

\[
\frac{\partial a_{\alpha}}{\partial u^\beta} = -\frac{\partial a_{\beta}}{\partial u^\alpha}
\]  

(2.68a)

Next

\[
\frac{\partial}{\partial u^\alpha} \left( a \cdot a \right) = \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}
\]  

(2.68b)

\[
\frac{\partial}{\partial u^\beta} \left( a \cdot a \right) = \frac{\partial g_{\alpha\gamma}}{\partial u^\alpha}
\]  

(2.68c)

\[
\frac{\partial}{\partial u^\gamma} \left( a \cdot a \right) = \frac{\partial g_{\beta\gamma}}{\partial u^\beta}
\]  

(2.68d)

Adding Eq. 2.68c and Eq. 2.68d and subtracting Eq. 2.68b while using Eq. 2.68a, one obtains

\[
\frac{\partial a_{\alpha}}{\partial u^\beta} \cdot a^\theta = g_{\alpha\beta}^\theta
\]  

(2.69)

where \( a^\theta \) are the contravariant surface base vectors satisfying

\[
a^\alpha \cdot a = \delta^\alpha_\beta
\]  

(2.70a)

and

\[
a^\theta = g^{\theta\alpha} a_{\alpha} \text{ etc.}
\]  

(2.70b)

As a caution, one must not hurriedly conclude an equation similar to Eq. 2.30b from Eq. 2.69. It must also be mentioned here that according to Eq. 2.70a, \( a^1 \) is orthogonal to \( a_2 \) and \( a^3 \) is orthogonal to \( a_1 \), but still \( a^1 \) and \( a^2 \) lie in the tangent plane to the surface.

### 2.4.2 Normal Curvature and the Second Fundamental Form

A plane containing the unit tangent vector \( t \) and the unit surface normal vector \( n \) at a point \( P \) of the surface cuts the surface in different curves when rotated about \( n \) as an axis. We refer to Figure 2.2, where the vectors \( t, n, k \), the curvature vector \( \hat{k} \), and another unit vector \( e \) in the tangent plane are shown.

Each curve obtained by rotating the \( t-n \) plane is called a normal section of the surface at \( P \). Since these curves belong both to the surface and also the embedding space, a study of the curvature properties of these curves also reveals the curvature and torsion properties of the surface itself.

We decompose the vector \( \hat{k} \) at \( P \) of \( C \), defined by Eq. 2.48, as

\[
\hat{k} = k + k \quad \text{as}
\]  

(2.71)
where the vector \( k_n \), is normal to the surface, and the vector \( k_e \) is tangent to the surface as shown in Figure 2.2. The vector \( k \) is called the normal curvature vector at the point, and it is directed either toward or against the direction of the surface normal \( n \). Thus

\[ k = n k_n \]  

(2.72)

where \( k_n \) is the normal curvature of the normal section of the surface, and is an algebraic number. To find a formula for \( k_n \) we consider the equation

\[ n \cdot t = 0 \]

and differentiate it with respect to \( s \), which yields

\[ k_n = \frac{-dn \cdot dr}{(ds)^2} \]  

(2.73)

Next we differentiate

\[ n \cdot a = 0 \]

with respect to \( u^\alpha \) and have

\[ \frac{\partial n}{\partial u^\alpha} \cdot a = -n \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} \]

Further

\[ dn = \frac{\partial n}{\partial u^\alpha} du^\alpha, \quad dr = a \cdot du^\beta \]
Eq. 2.73 yields

\[
k_n = \left( n \cdot \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} \right) \frac{du^\alpha du^\beta}{(ds)^2}
\]  

(2.74)

A set of new coefficients \( b_{\alpha\beta} \) are now defined as

\[
b_{\alpha\beta} = n \cdot \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta}
\]  

(2.75a)

\[
= - \frac{\partial n}{\partial u^\alpha} \cdot a^\beta
\]  

(2.75b)

\[
= \frac{1}{\sqrt{G_3}} a \left( - \frac{a}{2} \times \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} \right)
\]  

(2.75c)

Thus Eq. 2.74, beside having the form given in Eq. 2.73, can also be stated as

\[
k_n = \frac{b_{\alpha\beta} du^\alpha du^\beta}{(ds)^2}
\]  

(2.76a)

\[
= \frac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\mu\nu} du^\mu du^\nu}
\]  

(2.76b)

It is easy to see from Eq. 2.76a that

\[
\left( k_n d r + d n \right) \cdot d r = 0
\]

But \( dr \) is arbitrary, so that

\[
k_n d r + d n = 0
\]  

(2.77)

which is due to Rodrigues [1].

The form

\[
b_{\alpha\beta} du^\alpha du^\beta
\]

is called the “second fundamental form,” and \( b_{\alpha\beta} \) the coefficients of the second fundamental form. In expanded form, writing \( \xi = u^1, \eta = u^2 \), we have
Returning to the consideration of $k_n$ we note that from Eqs. 2.71 and 2.72 which on using Eq. 2.51a gives

$$n \cdot \hat{k} = k_n$$

which on using Eq. 2.51a gives

$$k_k = k \cos \phi$$

where

$$p \cdot n = \cos \phi$$

and $k$ is the curvature of the curve $C$. Introducing the radius of curvatures

$$\rho = 1/k, \; \rho_n = 1/k_n$$

we get from Eq. 2.79

$$\rho = \rho_n \cos \phi$$

which is due to Meusnier [1].

### 2.4.3 Principal Normal Curvatures

Let us introduce the directions

$$l = \frac{d\xi}{ds}, \; m = \frac{d\eta}{ds}$$

then Eq. 2.76a takes the form

$$k_n = b_{11}l^2 + 2b_{12}lm + b_{22}m^2$$

If only the direction

$$\lambda = \frac{d\eta}{d\xi}$$
is introduced, then

\[ k_n = \frac{\dot{b}_{11} + 2\dot{b}_{12} + \lambda^2 \dot{b}_{22}}{g_{11} + 2\dot{\lambda} g_{12} + \lambda^2 g_{22}} \]  

(2.82)

With the coefficients \( g_{\alpha\beta} \) and \( b_{\alpha\beta} \) as constants at a point, the quantity \( k_n \) is a function of \( \lambda \). The extremum values of \( k_n \) are obtained by

\[ \frac{dk_n}{d\lambda} = 0 \]

and the roots of this equation determine those directions for which the normal curvatures \( k_n \) assumes extreme values. These extreme values are called the principal normal curvatures at \( P \) of the surface, which we shall denote by \( k_I \) and \( k_{II} \). The corresponding directions \( \lambda \) are called the principal directions. Following the details given in [2], we obtain the following important equations for the sum and product of the principal curvatures:

\[ k_I + k_{II} = b_{\alpha\beta} g^{\alpha\beta} \]  

(2.83)

\[ k_I k_{II} = \frac{b}{G_3} \]  

(2.84)

where \( G_3 \) and \( b \) have been defined in Eqs. 2.60d and 2.78d, respectively. Here a few definitions are in order.

(i) **Lines of curvature:**

The line of curvature is a curve in a surface whose curvature at any point is either \( k_I \) or \( k_{II} \). The tangent to the line of curvature falls in the principal direction. The equations for the determination of the lines of curvature are obtained by differentiating Eq. 2.82 with respect to \( \lambda \) and setting the result equal to zero. Thus

\[ A\left( \frac{d\eta}{d\xi} \right)^2 + B \frac{d\eta}{d\xi} + C = 0 \]  

(2.85)

where

\[ A = b_{22} g_{12} - b_{12} g_{22} \]
\[ B = b_{22} g_{11} - b_{11} g_{22} \]
\[ C = b_{12} g_{11} - b_{11} g_{12} \]

Note that Eq. 2.85 is equivalent to two first-order ordinary differential equations, and their solutions define two families of curves in a surface which are the lines of curvature. Further, these curves are orthogonal. It is obvious from Eq. 2.85 that if \( A = 0 \), then \( d\xi = 0 \), and if \( C = 0 \) then \( d\eta = 0 \). Thus the curves \( \xi = \text{const} \) and \( \eta = \text{const} \) are the lines of curvature if \( A = 0 \) and \( C = 0 \).

In an actual computation if the coefficients of the first and second fundamental forms are known throughout the surface as functions of \( \xi \) and \( \eta \), and further the initial point \( \xi_0, \eta_0 \) is prescribed, then
the curves of curvature can be obtained by a numerical method, e.g., the Runge–Kutta method. If the curves $\xi$ and $\eta$ are themselves the curves of curvature, then as discussed above in these coordinates $g_{12} = 0$ and $b_{12} = 0$, and from Eq. 2.82,

$$k_n = b_{11} \left( \frac{d\xi}{ds} \right)^2 + b_{22} \left( \frac{d\eta}{ds} \right)^2$$

(2.86)
a formula due to Euler. The normal curvatures are then

$$k_i = \frac{b_{11}}{g_{11}} \text{ for } \eta = \text{const. (} \xi \text{ - curve)}$$

$$k_{il} = \frac{b_{22}}{g_{22}} \text{ for } \xi = \text{const. (} \eta \text{ - curve)}$$

(ii) **Asymptotic directions:**

Points on a surface where $k_n = 0$ give two directions, which from Eq. 2.82 are

$$\frac{dn}{d\xi} = \frac{-b_{12} \pm \sqrt{(b_{12})^2 - b_{11}b_{22}}}{b_{22}}$$

(iii) **Results for a surface of the form** $z = f(x, y)$:

When the equation of a surface is given in the form $z = f(x, y)$, then it is convenient to take

$$x = \xi, \ y = \eta, \ z = f(\xi, \eta)$$

Then

$$r = i\xi + j\eta + kf(\xi, \eta)$$
$$\quad a_1 = i + kf_x$$
$$\quad a_2 = j + kf_y$$

$$g_{11} = 1 + f_x^2, \ g_{12} = f_x f_y, \ g_{22} = 1 + f_y^2$$

$$G_3 = 1 + f_x^2 + f_y^2$$

$$n = \left( -i f_x - j f_y + k \right) \sqrt{G_3}$$

$$dA = \sqrt{G_3} \ dx \ dy, \ \text{element of area}$$

$$b_{11} = f_{xx} \int \sqrt{G_3}, \ b_{12} = f_{xy} \int \sqrt{G_3}, \ b_{22} = f_{yy} \int \sqrt{G_3}$$

As an example, for a monkey saddle

$$z = y^3 - 3yx^2$$

for which all the geometrical elements can be computed from Eq. 2.87.
(iv) Results for a body of revolution:

Let a curve \( z = f(x) \) in the plane \( y = 0 \) be rotated about the \( z \)-axis. The surface of revolution so generated has the parametric representation

\[
x = \xi \cos \eta, \quad y = \xi \sin \eta, \quad z = f(\xi)
\]

where \( \xi > 0 \) and \( \frac{df}{d\xi} = f' \) is bounded. For this case,

\[
a_i = i \cos \eta + j \sin \eta + k f'
\]

\[
a_j = \xi (i \sin \eta + j \cos \eta)
\]

\[
g_{11} = 1 + f'^2, \quad g_{12} = 0, \quad g_{22} = \xi^2, \quad G_3 = \xi^2 (1 + f^2)
\]

\[
n = \frac{1}{\sqrt{1 + f'^2}} \left( f' \cos \eta + j f' \sin \eta - k \right)
\]

\[
b_{11} = \frac{f''}{\sqrt{1 + f'^2}}, \quad b_{12} = 0, \quad b_{22} = \frac{\xi f'}{\sqrt{1 + f'^2}}
\]

Also referring to Eq. 2.93,

\[
Y_{11}^1 = \frac{f''}{1 + f^2}, \quad Y_{22}^1 = \frac{\xi}{1 + f^2}, \quad Y_{12}^1 = \frac{1}{\xi}
\]

and all other Christoffel symbols are zero.

As a particular case, for a cone

\[
\xi = r \sin \alpha, \quad f(\xi) = r \cos \alpha
\]

where \( r \) is the radial distance from the origin (apex of the cone) to a point on the cone’s surface, and \( \alpha \) is the angle made by \( r \) with the \( z \)-axis. Then

\[
x = r \sin \alpha \cos \eta, \quad y = r \sin \alpha \sin \eta, \quad z = r \cos \alpha
\]

which yields the equation of a cone:

\[
x^2 + y^2 = z^2 \tan^2 \alpha
\]

2.4.4 Mean and Gaussian Curvatures

The mean curvature \( K_m \) of a surface at a point is defined as

\[
k_m = \frac{1}{2}(k_I + k_H)
\]

(2.88a)

while the Gaussian or total curvature at a point is defined as

\[
K = k_I k_H
\]

(2.88b)
Surfaces for which \( K_m = 0 \) are called "minimal" surfaces, while surfaces for which \( K = 0 \) are called developable surfaces. The manner in which \( k_I \) and \( k_{II} \) have been obtained and the Gaussian curvature \( K \) has been formed suggests that \( K \) is an extrinsic property. In fact, \( K \) is an intrinsic property of a surface, that is, it depends only on the first fundamental form and on the derivatives of its coefficients [1, 2, 7].

2.4.5 Derivatives of the Surface Normal; Formulae of Weingarten

From the simple identity

\[
n \cdot n = 1
\]

one obtains by differentiation the following two equations:

\[
\frac{\partial n}{\partial u^\alpha} = 0, \quad \alpha = 1, 2
\]

These two equations suggest that \( \frac{\partial n}{\partial u^\alpha}, \ \alpha = 1, 2 \), lie in the tangent plane to the surface. Thus

\[
\frac{\partial n}{\partial u^1} = P \alpha + Q \alpha_{-1}^{\text{-2}}
\]

\[
\frac{\partial n}{\partial u^2} = R \alpha + S \alpha_{-1}^{\text{-2}}
\]

To find the coefficients \( P, Q, R, S \), we differentiate \( \eta \cdot q_1 = 0 \) with respect to the \( u^2 \) and \( \eta \cdot q_2 = 0 \) with respect to \( u^1 \). The solution of the four scalar equations yields [7],

\[
\frac{\partial n}{\partial u^\alpha} = -b_{\alpha \beta} g^{\beta \gamma} a_{-\gamma}, \quad \alpha = 1, 2
\]

Eq. 2.89 were obtained by Weingarten [2, 7], and provide the formulae for the partial derivatives of the surface normal vector with respect to the surface coordinates.

2.4.6 Formulae of Gauss

In \( E^3 \) the vectors \( q_1, q_2, \eta \) form a system of independent vectors. It should therefore be possible to express the first partial derivatives of a base vector in terms of the base vectors themselves. Based on the preceding developments, the logical outcome is to have

\[
\frac{\partial a}{\partial u^\beta} = \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} = \gamma_{\alpha \beta} a + n b_{\alpha \beta}
\]

As a check we note that the dot products of Eq. 2.90 with \( q^\alpha \) and \( \eta \) yield Eqs. 2.69 and 2.75a, respectively. Eq. 2.90 provides the formulae of Gauss for the second derivatives \( \partial^2 r / \partial u^\alpha \partial u^\beta \).

The coefficients of the second fundamental form \( b_{\alpha \beta} \) for a surface have already been defined in Eq. 2.75a. One can obtain a new formula for them by considering the Gauss' formulae, Eq. 2.90, and the space Christoffel symbols as stated in Eq. 2.32. In \( E^3 \) consider a surface defined by \( x^3 = \text{const.} \), and let \( x^1 = u^1 \) and \( x^2 = u^2 \). Then from Eq. 2.32,
\[
\frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} = \Gamma^1_{\alpha\beta} a + \Gamma^2_{\alpha\beta} a + \Gamma^3_{\alpha\beta} a ; x^3 = \text{const}
\]

Since both \( a_1 \) and \( a_2 \) have been evaluated at \( x^3 = \text{const.} \), taking the dot product with the unit surface normal vector \( \hat{n} \), one gets

\[
n \cdot \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} = \Gamma^3_{\alpha\beta} \left( n \cdot a \right)
\]

Writing

\[
n \cdot a = \lambda
\]

and comparing with Eq. 2.90, one obtains

\[
b_{ab} = \left( \lambda \Gamma^3_{ab} \right) x^3 = \text{const.}
\]

which can also be used to find the coefficients \( b_{ab} \), \[16\]. Thus the formulae of Gauss can also be stated as

\[
\frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} = Y^v_{ab} a + n \left( \Gamma^3_{ab} \lambda \right) x^3 = \text{const.}
\]

From Eq. 2.66, the expanded form of the surface Christoffel symbols for the surface \( x^3 = \text{const.} \) and with \( u^1 = \xi, u^2 = \eta \) are as follows:

\[
\begin{align*}
Y^1_{11} &= \left[ g_{22} \frac{\partial g_{11}}{\partial \xi} + g_{12} \left( \frac{\partial g_{11}}{\partial \eta} - 2 \frac{\partial g_{12}}{\partial \xi} \right) \right] / 2G_3 \\
Y^2_{22} &= \left[ g_{11} \frac{\partial g_{22}}{\partial \eta} + g_{12} \left( \frac{\partial g_{22}}{\partial \xi} - 2 \frac{\partial g_{12}}{\partial \eta} \right) \right] / 2G_3 \\
Y^1_{22} &= \left[ g_{22} \left( 2 \frac{\partial g_{11}}{\partial \eta} \frac{\partial g_{22}}{\partial \xi} \right) - g_{12} \frac{\partial g_{22}}{\partial \eta} \right] / 2G_3 \\
Y^2_{11} &= \left[ g_{11} \left( 2 \frac{\partial g_{12}}{\partial \xi} \frac{\partial g_{22}}{\partial \eta} \right) - g_{12} \frac{\partial g_{11}}{\partial \xi} \right] / 2G_3 \\
Y^1_{12} &= Y^2_{21} = \left[ g_{22} \frac{\partial g_{11}}{\partial \eta} - g_{12} \frac{\partial g_{22}}{\partial \xi} \right] / 2G_3 \\
Y^2_{12} &= Y^2_{21} = \left[ g_{11} \frac{\partial g_{22}}{\partial \xi} - g_{12} \frac{\partial g_{11}}{\partial \eta} \right] / 2G_3 \\
Y^1_{11} + Y^2_{12} &= \frac{1}{2G_3} \frac{\partial G_3}{\partial \xi} \\
Y^1_{12} + Y^2_{22} &= \frac{1}{2G_3} \frac{\partial G_3}{\partial \eta} \\
G_3 &= g_{11} g_{22} - \left( g_{12} \right)^2
\end{align*}
\]
2.4.7 Gauss–Codazzi Equations

Consider the identity

\[
\frac{\partial}{\partial u^\gamma} \left( \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} \right) = \frac{\partial}{\partial u^\mu} \left( \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} \right)
\]

for any choice of \( \alpha, \beta, \) and \( \gamma \). Using Eq. 2.90 and then Eq. 2.89, one obtains

\[
R_{\mu\alpha\gamma\beta} - \left( b_{\alpha\beta} b_{\lambda\mu} - b_{\alpha\gamma} b_{\beta\mu} \right) = 0 \tag{2.94}
\]

and

\[
\frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} + Y_{\alpha\beta}^\lambda b_{\delta\gamma} - Y_{\alpha\gamma}^\lambda b_{\beta\lambda} = 0 \tag{2.95}
\]

where \( R_{\mu\alpha\gamma\beta} \) is the two-dimensional Riemann curvature tensor, given as

\[
R_{\mu\alpha\gamma\beta} = g_{\mu\delta} \left( \frac{\partial Y_{\alpha\beta}^\delta}{\partial u^\gamma} - \frac{\partial Y_{\alpha\gamma}^\delta}{\partial u^\beta} + Y_{\alpha\beta}^\sigma Y_{\sigma\delta} - Y_{\alpha\gamma}^\sigma Y_{\beta\lambda} \right) \tag{2.96}
\]

Eq. 2.94 is called the equation of Gauss and is exhibited here in tensor form. In two dimensions, only four components are non-zero. That is

\[
R_{1212} = R_{2121} = b
\]

and

\[
R_{2112} = R_{1221} = -b
\]

where

\[
b = b_{11} b_{22} - (b_{12})^2
\]

The Gaussian curvature \( K \) is given by

\[
K = R_{1212} / G_3 \tag{2.97}
\]

On the other hand, Eq. 2.95 yields two equations: one for \( \alpha = 1, \beta = 1, \gamma = 2 \) and the other for \( \alpha = 2, \beta = 2, \gamma = 1 \). The resulting two equations are called the Codazzi or Codazzi–Mainardi equations.

2.4.8 Second-Order Differential Operator of Beltrami

First of all, it is of interest to note that Eqs. 2.35b and 2.36b for the covariant derivative and Eqs. 2.37a, 2.37b and 2.39 are all valid in any space including \( E^3 \), and are equally applicable to a surface that is
nothing but a two-dimensional non-Euclidean space. Thus the above-noted formulae for a surface are as follows:

\[
\begin{align*}
  u_\alpha^\beta &= \frac{\partial u^\alpha}{\partial u^\beta} + Y_{\alpha\beta} u^\gamma \\
  u_{\alpha,\beta} &= \frac{\partial u_{\alpha}}{\partial u^\beta} + Y_{\alpha\beta} g_{\gamma} \\
  \frac{\partial g_{\alpha\beta}}{\partial u^\gamma} &= -Y_{\delta\gamma} g^{\delta\alpha} - Y_{\beta\gamma} g^{\delta\alpha} \\
  \frac{\partial g_{\alpha\beta}}{\partial u^\gamma} &= -Y_{\delta\gamma} g^{\alpha\alpha} - Y_{\beta\gamma} g^{\alpha\alpha} \\
  Y^\gamma_{\delta} &= \frac{1}{2G_3} \frac{\partial G_3}{\partial u^\beta} \\
  &= \frac{\partial}{\partial u^\beta} \left( \ln \sqrt{G_3} \right)
\end{align*}
\]

The second-order differential operator of Beltrami when applied to a function \( \phi \) yields \[2\]

\[
\Delta_2 \phi = \frac{1}{\sqrt{G_3}} \frac{\partial}{\partial u^\alpha} \left( \sqrt{G_3} g^{\alpha\beta} \frac{\partial \phi}{\partial u^\beta} \right) 
\]

(2.99)

Suppose \( \phi = u^\delta \), a surface coordinate, then

\[
\Delta_2 u^\delta = \frac{1}{\sqrt{G_3}} \frac{\partial}{\partial u^\alpha} \left( \sqrt{G_3} g^{\alpha\delta} \right)
\]

(2.100)

Using the formulae given in Eq. 2.98, we get

\[
\Delta_2 u^\delta = -Y_{\alpha\beta} g^{\alpha\beta}
\]

(2.101a)

Note the exact similarity between Eqs. 2.44 and 2.100, and between Eqs. 2.34 and 2.101a. Using the formulae given in Eqs. 2.98, Eq. 2.99 becomes

\[
\Delta_2 \phi = g_{\alpha\beta} \left( \frac{\partial^2 \phi}{\partial u^\alpha \partial u^\beta} - Y_{\alpha\beta} \frac{\partial \phi}{\partial u^\gamma} \right)
\]

(2.101b)

or, by using Eq. 2.101a,

\[
\Delta_2 \phi = g^{\alpha\beta} \frac{\partial^2 \phi}{\partial u^\alpha \partial u^\beta} + (\Delta_2 u^\delta) \frac{\partial \phi}{\partial u^\delta}
\]

(2.101c)
2.4.9 Geodesic Curves in a Surface

The geodesic curves in a surface are defined in two ways [1]:

(i) Geodesics are curves in a surface that have zero geodesic curvature.
(ii) Geodesic curves are lines of shortest distance between points on a surface.

In the first definition, we must first obtain the formula for the geodesic curvature. Referring to Eq. 2.71 and Figure 2.2, we write the curvature vector of a curve

\[ \mathbf{k} = n k_n + e k_g \]  

(2.102)

where the unit vector \( e \) lies in the tangent plane to the surface. Refer to Figure 2.2. Note that

\[ e = n \times t \]

and

\[ k_g = e \cdot \mathbf{k} \]

\[ = e \left( n \times t \right) \cdot \frac{dt}{ds} \]

\[ = \left( n \times t \right) \cdot \frac{dt}{ds} \]

\[ = \left( t \times n \right) \cdot n \]  

(2.103a)

Further

\[ \frac{dt}{ds} = \frac{\partial a}{\partial u^\alpha} \frac{du^\alpha}{ds} + a \frac{d^2 u^\alpha}{ds^2} \]  

(2.103b)

Using the formulae of Gauss, Eq. 2.90, in Eq. 2.103b, putting the result in Eq. 2.103a, and writing \( u^1 = \xi \), \( u^2 = \eta \), we get after some simplification

\[ k_g \int \sqrt{G_3} Y_{11}^2 \left( \frac{d\xi}{ds} \right)^3 - Y_{12}^1 \left( \frac{d\eta}{ds} \right)^3 + \left( 2Y_{12}^2 - Y_{11}^1 \right) \left( \frac{d\xi}{ds} \right)^2 \frac{d\eta}{ds} \]

\[ - \left( 2Y_{12}^1 - Y_{22}^2 \right) \left( \frac{d\eta}{ds} \right)^2 \frac{d\xi}{ds} + \frac{d\xi}{ds} \frac{d^2 \eta}{ds} - \frac{d\eta}{ds} \frac{d^2 \xi}{ds} \]  

(2.104)

Eq. 2.104 is the formula for the geodesic curvature of a curve \( C \) in the surface with reference to the surface coordinates \( \xi, \eta \). Here \( s \) is the arc length along the curve \( C \). From Eq. 2.104, the geodesic curvature of the coordinate curve \( \eta \) or \( \xi = \text{const.} \) is

\[ \left( k_g \right)_{\xi=\text{const.}} = -\sqrt{G_3} Y_{12}^1 / g_{22}^{3/2} \]  

(2.105a)
and the geodesic curvature of the coordinate curve \( \xi \) or \( \eta = \text{const.} \) is

\[
(k_g)_{\eta=\text{const.}} = \frac{G_3 Y_{11}^2 \gamma^1}{g_{11}^{1/2}}
\] (2.105b)

Obviously if the \( \eta \)-curve is a geodesic then \( Y_{22}^1 = 0 \), while if the \( \xi \)-curve is a geodesic then \( Y_{11}^2 = 0 \). The differential equation for the geodesic curve is obtained from Eq. 2.104 by putting \( k_g = 0 \). For brevity, writing

\[
\begin{align*}
\xi' &= \frac{d\xi}{ds}, \\
\eta' &= \frac{d\eta}{ds}
\end{align*}
\]

and using

\[
\xi'\eta'' - \eta'\xi'' = \xi'^2 \frac{d}{ds} \left( \frac{\eta''}{\xi'} \right) = \xi'^2 \frac{d}{ds} \left[ \frac{d\eta}{d\xi} \right]
\]

we get

\[
\frac{d^2\eta}{d\xi^2} - Y_{22}^1 \left( \frac{d\eta}{d\xi} \right)^3 - (2Y_{12}^1 - Y_{22}^2) \left( \frac{d\eta}{d\xi} \right)^2 + (2Y_{12}^1 - Y_{11}^1) \frac{d\eta}{d\xi} + Y_{11}^2 = 0
\] (2.106)

By solving Eq. 2.106 under the initial conditions

\[
\begin{align*}
\eta(\xi_0), \text{ (point); and } \left( \frac{d\eta}{d\xi} \right)_{\xi=\xi_0}, \text{ (direction)}
\end{align*}
\]

a unique geodesic can be obtained. According to [3], a geodesic can be found to pass through any given point and have any given direction at that point. If the Christoffel symbols are known for all points of a surface in terms of the surface coordinates \( \xi, \eta \), then a numerical method, e.g., the Runge–Kutta method, can be used to solve Eq. 2.106.

In \( E^3 \) a straight line is the shortest distance between two points. A generalization of this concept to Riemannian or non-Euclidean spaces can be accomplished by using the integral of Eq. 2.46 and applying the Euler–Langrange equations. The end result (refer to [2]) is that the intrinsic derivative (Eq. 2.41) applied to the contravariant components of the unit tangent vector \( t^\alpha \) with the parameter \( t \) replaced by the arc length \( s \) is zero. That is,

\[
\frac{\delta}{ds} \left( \frac{du^\gamma}{ds} \right) = 0
\]

which yields

\[
\frac{d^2u^\alpha}{ds^2} + Y_{\beta\alpha} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = 0, \; \alpha = 1,2
\] (2.107)
The two second-order ordinary differential equations from Eq. 2.107 can be solved simultaneously to yield the geodesic curves \( u^1 = u^1(s) \), \( u^2 = u^2(s) \) by specifying the initial conditions. Alternatively, writing \( u^1 = \xi, u^2 = \eta \) and

\[
\frac{d\eta}{d\xi} = \frac{d\eta}{ds} \cdot \frac{ds}{d\xi} = \frac{\eta'}{\xi'} \\
\frac{d^2\eta}{d\xi^2} = \frac{\eta''}{\xi''} - \frac{\xi''\eta'}{\xi'^3}
\]

and using the two equations from Eq. 2.107, one obtains Eq. 2.106.

### 2.4.10 Geodesic Torsion

The torsion of the geodesic of a surface is called the geodesic torsion and is denoted by \( \tau_g \). Before we proceed further, it is important to note that the basic triads of vectors for space curves is \( (t, p, b) \) and for the surface curves is \( (t, e, n) \). It can be proved (refer to [2]) that for a surface geodesic the unit normal \( n \) to a surface at a point is equal to the principal normal \( p \) of the surface geodesic at the same point, i.e., \( p = n \). Thus from Eq. 2.50,

\[
b = t \times n
\]

and from Eq. 2.51c,

\[
\frac{db}{ds} = -\tau_g n
\]

Thus

\[
\frac{dt}{ds} \times n + t \times \frac{dn}{ds} = -\tau_g n
\]

The first term is zero, since \( \hat{k} \) is parallel to \( n \), and we obtain

\[
\tau_g = n \left( \frac{dn}{ds} \times t \right)
\]

To establish a relation between the torsion \( \tau \) of a curve \( C \) lying on a surface and the torsion of the geodesic \( \tau_g \) which touches \( C \) at the point \( P \), we consider Eq. 2.102 and write it as

\[
k_p = nk_n + ek_g
\]

where \( k \) is the curvature of the curve \( C \) and \( k_g \) is the geodesic (tangential) curvature of the surface at \( P \). Further, using the relation

\[
k_g = k \sin \phi
\]
from Figure 2.2 and Eq. 2.79, we get

\[ p = n \cos \phi + e \sin \phi \]  

(2.109)

On differentiating Eq. 2.109 with respect to \( s \), using Eq. 2.51b, and taking the dot product with \( \eta \), we obtain

\[ \tau (b \cdot n) = n \cdot \frac{de}{ds} \sin \phi - \frac{d\phi}{ds} \sin \phi \]

Differentiating

\[ e = n \times t \]

using \( b \cdot \eta = \sin \phi \neq 0 \), and Eq. 2.108, we get

\[ \tau_g = \tau + \frac{d\phi}{ds} \]  

(2.110)

### 2.5 Elliptic Equations for Grid Generation

In this section we shall develop the elliptic equations for grid generation, or numerical coordinate mapping, in both the Euclidean and non-Euclidean spaces. The mathematical apparatus to achieve this aim has already been developed in Sections 2.2 through 2.4. In this regard the following two important points should be noted.

(i) Depending on the number of space dimensions, one has to choose a set of “grid or coordinate generators,” which form a sort of constraints on the variables of computational or logical space.

(ii) The resulting grid generation equations should be obtained in a form in which the computational space variables appear as the independent variables rather than the dependent variables.

#### 2.5.1 Elliptic Grid Equations in Flat Spaces

First by setting \( \phi = r \) in Eq. 2.45a and noting that \( r = \sum_{m=1}^{n} \gamma_{m}^{i} x_{m} \) so that its Laplacian is zero, we have

\[
g^{ij} \left( \frac{\partial^{2} r}{\partial x^{i} \partial x^{j}} - \Gamma^{k}_{ij} \frac{\partial r}{\partial x^{k}} \right) = 0
\]

Using Eq. 2.34, we get

\[
g^{ij} \frac{\partial^{2} r}{\partial x^{i} \partial x^{j}} + \left( \nabla^{2} x^{k} \right) \frac{\partial r}{\partial x^{k}} = 0
\]  

(2.111)

If we now take the grid generators as a set of Poisson equations, i.e.,

\[
\nabla^{2} x^{k} = p^{k}
\]  

(2.112)
where $P^k$ are arbitrary functions of the coordinates $x^i$, then from the identity shown as Eq. 2.111 a deterministic set of equations is obtained, which is

$$D r + g P^k \frac{\partial r}{\partial x^k} = 0$$

(2.113)

where $D$ is a second-order differential operator defined as

$$D = g g^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$$

Writing $r = \ell_m x_m$, where $x_m(x^1,x^2,x^3)$ with $m = 1, 2, 3$, one can readily write three coupled quasilinear partial differential equations for $x_1, x_2, x_3$ from Eq. 2.113. Writing $x^1 = \xi$, $x^2 = \eta$, $x^3 = \zeta$, denoting a partial derivative by a variable subscript, and using Eq. 2.22, the operator $D$ is written as

$$D = G_i \partial_{\xi} + G_2 \partial_{\eta} + G_3 \partial_{\zeta} + 2 G_4 \partial_{\xi \eta} + 2 G_5 \partial_{\xi \zeta} + 2 G_6 \partial_{\eta \zeta}$$

(2.114a)

In two dimensions there is no dependence on $z$ and $g_{33} = 1$, so that

$$D = g_{22} \partial_{\xi} - 2 g_{12} \partial_{\eta} + g_{11} \partial_{\eta}$$

(2.114b)

and the two equations for $x_1 = x, x_2 = y$, from Eq. 2.113 are

$$g_{22} x_{\xi} - 2 g_{12} x_{\eta} + g_{11} y_{\eta} + g \left( P^1 x_{\xi} + P^2 x_{\eta} \right) = 0$$

(2.115a)

$$g_{22} y_{\xi} - 2 g_{12} y_{\eta} + g_{11} y_{\eta} + g \left( P^1 y_{\xi} + P^2 y_{\eta} \right) = 0$$

(2.115b)

A more general choice for $P^k$ is to take it as [15–17]

$$P^k = g^{ij} P^k_{ij}$$

(2.116)

where $P^k_{ij} = P^k_{ji}$ are arbitrary functions. As an example, with this choice the $P^i$ and $P^j$ appearing in Eqs. 2.115 become

$$P^k = \left( g_{22} P^k_{11} - 2 g_{12} P^k_{12} + g_{11} P^k_{22} \right)/g, \, k = 1, 2$$

(2.117)

Note that the $g$ appearing in Eqs. 2.115 and 2.117 is

$$g = g_{11} g_{12} - (g_{12})^2$$

With the choice of Eq. 2.116, Eq. 2.113 becomes
Either Eq. 2.113 or Eq. 2.118 forms the basic coordinate generation equations of the elliptic type in Euclidean spaces. For engineering and applied sciences, usually the Euclidean spaces of two \((E^2)\) or three \((E^3)\) dimensions are needed. In all cases these equations are quasilinear and are solved numerically under the Dirichlet or mixed Dirichlet and Neumann boundary conditions. Note that both Eqs. 2.113 and 2.118 are elliptic partial differential equations in which the independent variables are \(x^i\) or \(\xi, \eta, \zeta\), and the dependent variables are the rectangular Cartesian coordinates \(r = (x_m) = (x, y, z)\).

### 2.5.1.1 Coordinate Transformation

Let \(x^i\) be another coordinate system such that

\[
\tilde{x}^i = f^i(x^1, x^2, x^3), \quad i = 1, 2, 3
\]

A transformation from one coordinate system to another is said to be admissible if the transformation Jacobian \(J \neq 0\), where

\[
J = \det \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right)
\]  

(2.119a)

Under the condition \(J \neq 0\), the inverse transformation

\[
x^i = \phi^i(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)
\]

exists and

\[
\tilde{J} = \det \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right)
\]  

(2.119b)

where \(\tilde{J} \neq 0\).

The theory of coordinate transformation plays two key roles in grid generation. First, if the coordinates \(x^i\) are considered, then Eq. 2.118 takes the form

\[
D r + g g^i j P^k_0 \frac{\partial r}{\partial x^k} = 0
\]  

(2.120)

How are the control system function \(P^i_0\) and \(\bar{P}^k_{ij}\) related? An answer to this question may provide a significant advancement towards the problem of adaptivity. For details on the relationships between \(P^i_0\) and \(\bar{P}^k_{ij}\) refer to [15] and [23]. Second, the consideration of coordinate transformation leads one to the generating equations in which the dependent variables are not the rectangular Cartesian coordinates. For example, in some problems the dependent variables may be cylindrical coordinates.

Before proceeding on the second topic it will be helpful to summarize some basic transformation formulae. Refer to [2, 7], etc.
Using Eq. 2.121c in Eq. 2.121d, we get

\[
\frac{\partial^2 \bar{x}^p}{\partial \bar{x}^k \partial \bar{x}^n} = \Gamma_{kn}^i \frac{\partial \bar{x}^p}{\partial \bar{x}^s} - \Gamma_{rt}^p \frac{\partial \bar{x}^r}{\partial \bar{x}^s} \frac{\partial \bar{x}^t}{\partial \bar{x}^n} \tag{2.121e}
\]

Inner multiplication yields

\[
\frac{\partial^2 x^s}{\partial x^m \partial x^n} = -\frac{\partial^2 x^p}{\partial x^k \partial x^n} \frac{\partial x^s}{\partial x^p} \frac{\partial x^k}{\partial x^m} \frac{\partial x^r}{\partial x^n} \tag{2.121f}
\]

Eq. 2.121e, 2.121f provide the formulae for the second derivatives. The first partial derivatives of \(x^i\) with respect to \(\bar{x}^j\) are given by

\[
\frac{\partial x^i}{\partial \bar{x}^j} = \frac{C^i_j}{J} \tag{2.121g}
\]

\[
C^i_j = \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial x^k}{\partial x^n} \frac{\partial x^r}{\partial x^n} \frac{\partial x^k}{\partial x^s} \tag{2.121h}
\]

where \((i, s, n)\) and \((j, r, k)\) are cyclic permutations of \((1, 2, 3)\), and \(J\) is defined by Eq. 2.119a.

According to Eq. 2.34 the Laplacian of the coordinates \(\bar{x}^i\) is

\[
\nabla^2 \bar{x}^i = -g^{ij} \Gamma_{ij}^s \tag{2.122}
\]

and

\[
\nabla^2 x^k = -g^{ij} \Gamma_{ij}^s = g^{ij} p^k = p^k \tag{2.123}
\]

Thus writing \(\phi = \bar{x}^i\) in Eq. 2.45b and using Eqs. 2.122 and 2.123, we get

\[
g^{ij} = \frac{\partial^2 \bar{x}^s}{\partial x^i \partial x^j} + p^k \frac{\partial \bar{x}^s}{\partial x^k} = \bar{g}^{ij} \Gamma_{ij}^s \tag{2.124}
\]
Writing

\[ g^{ij} = g^{mn} \frac{\partial x^i}{\partial x^m} \frac{\partial x^j}{\partial x^n} \]

in Eq. 2.124 and using Eq. 2.121g, we get

\[ C_m^i C_n^j \frac{\partial^2 \tilde{x}^i}{\partial \tilde{x}^j \partial \tilde{x}^k} + J^2 \Gamma^k \frac{\partial \tilde{x}^i}{\partial \tilde{x}^k} = -J^2 g^{ij} \Gamma^s_{ij} \]  

(2.125)

For prescribed functions \( P^k \), the set of Eq. 2.125 generates the \( \tilde{x}^i \) coordinates as functions of \( x^i \) coordinates. Here \( \tilde{x}^i \) can be either rectangular Cartesian or any other coordinate system, e.g., cylindrical. Note that if \( \tilde{x}^i \) are rectangular Cartesian coordinates, then

\[ C_m^i C_n^j \frac{\partial^2 \tilde{x}^i}{\partial \tilde{x}^j \partial \tilde{x}^k} + J^2 \Gamma^k \frac{\partial \tilde{x}^i}{\partial \tilde{x}^k} = 0 \]

so that Eq. 2.125 becomes Eq. 2.113.

2.5.1.2 Non-Steady Coordinates

There are many situations in which the curvilinear coordinates are changing with time. This occurs mostly in problems where the coordinates move in an attempt to produce an adaptive solution. For a review of the time-dependent coordinates the reader is referred to [22]. For our present purposes we consider one possible grid generator to obtain time-dependent coordinates.

Basically a time-dependent coordinate system \( x^i \) is stated as

\[ x^i = x^i(\tau, t), \quad i = 1, 2, 3 \]  

(2.126a)

\[ \tau = t \]  

(2.126b)

and its inverse as

\[ \tau = \tau(x^i, \tau) \]  

(2.127a)

\[ t = \tau \]  

(2.127b)

From [22], we have the result

\[ \frac{\partial r}{\partial \tau} = -\frac{\partial r}{\partial x^k} \frac{\partial x^k}{\partial \tau} \]  

(2.128)
Suppose for time-dependent coordinates we change the grid generator, Eq. 2.113, to the form

$$\nabla^2 \chi^k = P^k + \phi \frac{\partial \chi^k}{\partial t}$$  \hspace{1cm} (2.129)

where $\phi = \phi(x^i)$. One may choose $\phi = c/g$, or, $\phi = c$, where $c$ is a constant. Substitution of Eq. 2.129 in Eq. 2.111 with $\phi = c$ and using Eq. 2.128 yields

$$g \frac{\partial r}{\partial \sigma} = D r + g P^k \frac{\partial r}{\partial x^k}$$  \hspace{1cm} (2.130)

where $\sigma = \tau/c$ and the operator $D$ is same as used in Eq. 2.113. Eq. 2.130 is parabolic in $\sigma$ and may be used to proceed in stepwise fashion from some initial time. It must, however, be noted that the success of the grid generator, Eq. 2.129, depends upon a proper choice of the control functions $P^k$ or $P^i_j$, if the form of Eq. 2.116 is used. The proper choice of the control functions depends on the physical problem. Much work in this area remains to be done.

### 2.5.1.3 Nonelliptic Grid Generation

Besides the elliptic grid generation methodology as discussed in the preceding subsections, which gives the smoothest grid lines, many authors have used the parabolic and hyperbolic equation methodologies. In the hyperbolic grid generation as developed in [24] the grid generators are formed of the following three equations:

$$g_{13} = 0, \ g_{23} = 0, \ \sqrt{g} = \Delta V$$  \hspace{1cm} (2.131)

where $\Delta V$ is a prescribed cell volume. One may take a certain distribution of $x^i$ and $x^3$ at the surface $x^3 = \text{const.}$ and march along the $x^3$ direction. Efficient numerical schemes can be used if Eq. 2.131 are combined as a set of simultaneous first-order equations. It must, however, be noted that Eqs. 2.131 are not invariant to a coordinate transformation.

### 2.5.2 Elliptic Grid Equations in Curved Surfaces

The basic formulation of the elliptic grid generation equations for a curved surface, forming a two-dimensional Riemannian space, is available in [15–18], and [25]. Here we summarize the salient features of the equations with the intent of establishing the fact that the proposed equations are not the result of any sort of simplifying assumptions. (In this regard, readers are referred to [26].) Further, every coordinate system in a surface must satisfy the proposed equations irrespective of the method used to obtain them.

We consider a curved surface embedded in $E^3$ and use the formulae of Gauss as given in Equation 2.90. Inner multiplication of Equation 2.90 by $g^{ab}$ while using Eqs. 2.83 and 2.101a results in having

$$g^{ab} \frac{\partial^2 r}{\partial u^a \partial u^b} + \left( \Delta_2 u^\delta \right) \frac{\partial r}{\partial u^\delta} = n(k_1 + k_H)$$  \hspace{1cm} (2.132)

From Eq. 2.101c we note that by setting $\phi = \varepsilon$, the left-hand side of Eq. 2.132 can be written as $\Delta_2 \varepsilon$. Thus

$$\Delta_2 \varepsilon = n(k_1 + k_H)$$  \hspace{1cm} (2.133)

where in both Eqs. 2.132 and 2.133 $n$ is the surface unit normal vector. Also by using Eq. 2.99 we have
\[ \nabla^2 r = \frac{1}{\sqrt{G_3}} \frac{\partial}{\partial u^\alpha} \left( \sqrt{G_3} g^{\alpha\beta} \frac{\partial r}{\partial u^\beta} \right) \quad (2.134) \]

We will return to Eqs. 2.133 and 2.134 subsequently. First, in Eq. 2.132 writing \( x^1 = \xi \), \( x^2 = \eta \), and

\[ \Delta_2 \xi = -g^{\alpha\beta} \nabla^1_{\alpha\beta} = P \tag{2.135a} \]
\[ \Delta_2 \eta = -g^{\alpha\beta} \nabla^2_{\alpha\beta} = Q \tag{2.135b} \]

while using the operator \( D \) defined as

\[ D = G_3 g^{\alpha\beta} \frac{\partial^2}{\partial u^\alpha \partial u^\beta} \]
\[ = g_{22} \frac{\partial^2}{\partial \xi^2} - 2g_{12} \frac{\partial}{\partial \xi} + g_{11} \frac{\partial}{\partial \eta} \]

we get

\[ Dr + G_3 \begin{pmatrix} Pr & Qr \\ -\xi & -\eta \end{pmatrix} = n R \tag{2.137} \]

where

\[ R = G_3 \left( k_I + k_H \right) = g_{22} b_{11} - 2g_{12} b_{12} + g_{11} b_{22} \tag{2.138} \]

Eq. 2.137 is a deterministic equation for grid generation if the control functions \( P \) and \( Q \), which are the Beltramians of \( \xi \) and \( \eta \), respectively, given in Eq. 2.135, are prescribed. The three scalar equations from Eq. 2.137 are

\[ Dx + G_3 \begin{pmatrix} P \xi & Q \eta \end{pmatrix} = XR \tag{2.139a} \]
\[ Dy + G_3 \begin{pmatrix} P \eta & Q \eta \end{pmatrix} = YR \tag{2.139b} \]
\[ Dz + G_3 \begin{pmatrix} P \xi & Q \eta \end{pmatrix} = ZR \tag{2.139c} \]

where \( n = (X, Y, Z) \).

For prescribed \( P \) and \( Q \), which may be chosen as zero, the set of elliptic equations stated in Eq. 2.139 form a model for surface coordinate generation. Looking back we note that the basis of these equations are the formulae of Gauss. To check whether the same equations can be obtained by using the formulae of Weingarten stated in Eq. 2.89 we proceed from Eq. 2.134. First we use the easily verifiable identity

\[ g^{\alpha\beta} a_{-\alpha} = e^{\alpha\delta} a_{-\delta} \times n \]
in Eq. 2.134. Here

\[ \varepsilon^{11} = 0, \; \varepsilon^{12} = \sqrt{G_3}, \; \varepsilon^{21} = -1, \; \varepsilon^{22} = 0 \]

and as before

\[ a - \beta = \frac{\partial r}{\partial u^\alpha} \]

Thus

\[ \Delta_2 r = \frac{1}{\sqrt{G_3}} \frac{\partial}{\partial u^\alpha} \left( \sqrt{G_3} \varepsilon^{\alpha \delta} a - \delta \right) \]

Opening the differentiation and using Eq. 2.89 along with the definition of given in Eq. 2.64, we obtain

\[ \Delta_2 r = n b_{\alpha \beta} g^{\alpha \beta} \]

\[ = n (k_I + k_{II}) \]

which is precisely Eq. 2.133 or Eq. 2.132. From this analysis we conclude that the proposed set of equations, i.e., Eq. 2.132, satisfies both the formulae of Gauss and Weingarten. In summary, we may state the following:

(i) The solution of the proposed equations automatically satisfies the formulae of Gauss and Weingarten.

(ii) When the curved surface degenerates to a plane \( z = \text{const.} \), then the proposed equations reduce to the elliptic coordinate generation equation given as Eq. 2.115. In this situation the Beltrami operator reduces to the Laplace operator, i.e.,

\[ \Delta_2 \xi = \nabla^2 \xi, \; \Delta_2 \eta = \nabla^2 \eta, \]

The key term in the solution of Eq. 2.139 is the term \( k_I + k_{II} \) appearing on the right-hand side. For a given surface if this term can be expressed as a function of \( x, y, z \), then there is no difficulty in solving the system of equations. Suppose the equation of the surface is given as \( F(x, y, z) = 0 \), then from [17],

\[ k_I + k_{II} \left[ \left( F_y^2 + F_z^2 \right) 2 F_x F_z \frac{F_x F_{xx} - F_z^2 F_{xx} - F_x^2 F_{zz} }{2 F_x F_z \left( F_z^2 F_{xy} + F_y F_{xz} - F_z F_x F_{xz} - F_y F_z F_{xz} \right)} \right] + \left( F_x^2 + F_z^2 \right) 2 F_y F_z \frac{F_y^2 F_{xy} - F_z^2 F_{xy} - F_y^2 F_{yz} }{2 F_y F_z \left( F_z^2 F_{xy} + F_y F_{xz} - F_z F_x F_{xz} - F_y F_z F_{xz} \right)} \]

\[ + \left( F_x^2 + F_z^2 \right) \left( 2 F_y F_z F_{xy} - F_z^2 F_{xy} - F_y^2 F_{yz} \right) / P^3 F_x^2, F_z \neq 0 \]  

(2.140)

where

\[ P^2 = F_x^2 + F_y^2 + F_z^2 \]
If \( F_z = 0 \), then a cyclic interchange of the subscripts will yield a formula in which \( F_z \) does not appear in the denominator. Thus we see that the whole problem of coordinate generation in a surface through Eq. 2.139 depends on the availability of the surface equation \( F(x, y, z) = 0 \). Numerical solutions of Eq. 2.139 have been carried out for various body shapes, including the fuselage of an airplane [25]. Here the function \( F(x, y, z) = 0 \) was obtained by a least square fit on the available data. As an example, Figure 2.3 shows the distribution of coordinate curves on a hyperbolic paraboloidal shell.

To alleviate the problem of fitting the function \( F(x, y, z) = 0 \), another set of equations can be obtained from Eq. 2.139. The basic philosophy here is to introduce an intermediate transformation \((u, \nu)\) between \( E^3 \) and \((\xi, \eta)\), as shown in Figure 2.4.

Let \( u \) and \( \nu \) be the parametric curves in a surface in which the curvilinear coordinates \( \xi \) and \( \eta \) are to be generated. Introducing

\[
\bar{g}_{11} = r \cdot r \cdot \bar{g}_{12} = r \cdot r \cdot \bar{g}_{22} = r \cdot r \cdot \bar{g}_{22} = r \cdot r
\]

\[
\bar{G}_3 = \bar{G}_{11} \bar{G}_{22} - (\bar{G}_{12})^2, J_3 = u_{\xi \eta} - u_{\eta \xi}
\]

then from the expressions such as

\[
r_{-\xi} = r_{-\xi} u_{\xi} + r_{-\eta} v_{\xi}, r_{-\eta} = r_{-\eta} u_{\eta} + r_{-\eta} v_{\eta}
\]

and simple algebraic manipulations, Eq. 2.137 yields the following two equations.

\[
a u_{\xi \xi} - 2 b u_{\xi \eta} + c u_{\eta \eta} + J_3^2 \left( P u_{\xi} + Q u_{\eta} \right) = J_3^2 \bar{G}_3 u
\]  

(2.141a)

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where

\[ a = g_{22} / G_3, \quad b = g_{12} / G_3, \quad c = g_{11} / G_3 \]

and

\[
\begin{align*}
\bar{\Delta}_2 u &= \frac{1}{\sqrt{G_3}} \left[ \frac{\partial}{\partial u} \left( \frac{\bar{g}_{22}}{\sqrt{G_3}} \right) - \frac{\partial}{\partial v} \left( \frac{\bar{g}_{12}}{\sqrt{G_3}} \right) \right] \\
\bar{\Delta}_2 v &= \frac{1}{\sqrt{G_3}} \left[ \frac{\partial}{\partial v} \left( \frac{\bar{g}_{11}}{\sqrt{G_3}} \right) - \frac{\partial}{\partial u} \left( \frac{\bar{g}_{12}}{\sqrt{G_3}} \right) \right]
\end{align*}
\]

Eqs. 2.141 were also obtained independently in [27] and recently in [28] by using the Beltrami equations of quasiconformal mapping. Nevertheless, the simple conclusion remains that Eqs. 2.141 are a direct outcome of Eq. 2.137.

### 2.5.2.1 Transformation of the Surface Coordinates

Let \( \bar{\mathbf{R}}^\alpha = f^\alpha(u^1, u^2) \) be an admissible coordinate transformation in a surface. It is a matter of direct verification that

\[ n(u^1, u^2) = \bar{n}(\bar{u}^1, \bar{u}^2), \text{ invariant} \]

and

\[ k_I + k_{II} = \bar{k}_I + \bar{k}_{II}, \text{ invariant} \]

Using these and other derivative transformations, it can be shown that Eq. 2.132 transforms to

\[
\bar{g}^{\alpha\beta} \frac{\partial^2 \bar{r}}{\partial \bar{u}^\alpha \partial \bar{u}^\beta} + \left( \bar{\Delta}_2 \bar{\mathbf{R}}^\delta \right) \frac{\partial \bar{r}}{\partial \bar{u}^\delta} = \bar{n}(\bar{F}_1 + \bar{F}_{II})
\]

where

\[ \bar{\Delta}_2 \bar{u}^\delta = \bar{g}^{\alpha\delta} \bar{g}_{\alpha\beta} = \bar{g}^{\alpha\delta} \bar{P}_{\alpha\beta} \]

Similarly

\[ \Delta_2 \bar{r} = \bar{\Delta}_2 \bar{r} \]

The above analysis shows that Eq. 2.132 is form-invariant to coordinate transformation. The same result was obtained previously with regard to Eq. 2.118. How are the control functions \( P_{\alpha\beta}^\delta \) and \( \bar{P}_{\alpha\beta}^\delta \) related? An answer to this question is similar to the one addressed in [23] and is given in [17, Appendix A]. If
initially a harmonic coordinate system is chosen [29], then a recursive relation gives the subsequent surface coordinate control functions.

2.5.2.2 The Fundamental Theorem of Surface Theory

The fundamental theorem of surface theory proves the existence of a surface if the coefficients of the first and the second fundamental forms satisfy certain conditions. Referring to [1] the statement of the theorem is as follows: “If \( g_{\alpha \beta} \) and \( b_{\alpha \beta} \) are given functions of \( u^\delta \), sufficiently differentiable, which satisfy the Gauss and Codazzi equations as given in Eqs. 2.94 and 2.95, respectively and \( G_3 \neq 0 \), then there exists a surface that is uniquely determined except for its position in space.” The demonstration of this theorem consists in showing that the formulae of Gauss and Weingarten as given in Eqs. 2.90 and 2.89 respectively have to be solved under proper conditions. It may be noted that Eqs. 2.89 and 2.90 are 5 vector equations that yield 15 scalar equations, and the proper conditions are

\[
\begin{align*}
n \cdot n &= 1, \quad a \cdot n = 0, \quad \alpha = 1, 2 \\
-\alpha \cdot \alpha &= g_{\alpha \beta}, \quad \alpha, \beta = 1, 2 \\
\frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} &= b_{\alpha \beta}, \quad \alpha, \beta = 1, 2
\end{align*}
\]

The above statement poses an elaborate scheme and is quite involved for practical computations if one wants to generate a surface based on a knowledge of \( g_{\alpha \beta} \) and \( b_{\alpha \beta} \). A restatement of the fundamental theorem of surface theory is now possible because Eq. 2.132 already satisfies Eqs. 2.89 and 2.90. Thus, a restatement of the theorem is as follows: “If the coefficient \( g_{\alpha \beta} \) and \( b_{\alpha \beta} \) of the first and second fundamental forms have been given that satisfy the Gauss and Codazzi equations (Eqs. 2.94, 2.95), then a surface can be generated by solving only one vector equation (Eq. 2.132) to within an arbitrary position in space.” This theorem has been checked numerically for a number of cases [30].

2.5.2.3 Time-Dependent Surface Coordinates

If in a given surface the coordinates are time-dependent, then we take the “grid generator” similar to Eq. 2.129 with \( \phi = c \) as

\[
\Delta^2 u^\delta = g^{\alpha \beta} p_{\alpha \beta}^\delta, \quad + c \frac{\partial u^\delta}{\partial t}, \quad \delta = 1, 2
\]  

(2.143)

Realizing that the surface is defined by \( x^3 = \text{const.} \), the resulting surface grid generation equation becomes

\[
G_3 \frac{\partial r}{\partial \sigma} = \frac{\partial r}{\partial \tau} + \left( P r + Q r \right) - n R
\]  

(2.144)

where \( \sigma = \tau/c \) and all other quantities are similar to those given in Eq. 2.137. The choice \( \phi = c/G_3 \) has been used to generate the surface coordinates in a fixed surface by parametric stepping and using a spectral technique [31].

2.5.2.4 Coordinate Generation Equations in a Hypersurface

In the course of an effort to extend the fundamental basis of Eq. 2.132 we have considered an extension of the embedding space \( E^3 \) to a Riemannian-4 \((M^4)\) space. In \( M^4 \) let the local coordinates be \( x^i, i = 1, \ldots, 4 \) and let \( S \) be an immersed hypersurface of local coordinates \( \xi^\alpha, \alpha = 1, \ldots, 3 \). In the ensuing analysis, a comma preceding an index denotes a partial derivative. From
we note that $x^i_\alpha$ are the tangent vectors. Here, and in what follows, a comma preceding an index will denote a partial derivative while a semicolon will denote a covariant derivative. Further $g_{ij}$ and $a_{\alpha\beta}$ are the covariant metric tenors and $\Gamma^i_{jk}$ and $\Upsilon^\alpha_{\beta\gamma}$ are the Christoffel symbols in $M^4$ and $S$, respectively. The metric coefficients are related as

$$a_{\alpha\beta} = g_{ij} x^i_\alpha x^j_\beta$$

(2.145a)

$$a_{\alpha\beta} = g^{mn} a_{\alpha\beta}$$

(2.145b)

Let $a^i_\alpha$ be a contravariant vector in $M^4$ and a covariant vector in $S$, then from [2], the covariant derivative of $a^i_\alpha$ in $S$ is given by

$$a^i_{\alpha;\beta} = x^i_{\alpha,\beta} + a^r_\alpha \Gamma^i_{rk} x^k_\beta - a^i_\gamma \Upsilon^\gamma_{\alpha\beta}$$

(2.145c)

Replacing $a^i_\alpha$ by $x^i_\alpha$ in Eq. 2.145c, we get

$$x^j_{\alpha;\beta} = x^j_{\alpha,\beta} + \Gamma^j_{ik} x^i_\alpha x^k_\beta - \Upsilon^\gamma_{\alpha\beta} x^i_\gamma$$

(2.146)

From [2], the formulae of Gauss in a Riemannian manifold are

$$x^i_{\alpha;\beta} = b_{\alpha\beta} n^i$$

(2.147)

and the formula of Weingarten is

$$n^i_\beta = -b_{\alpha\beta} a_{\alpha\gamma} x^k_\gamma - \Gamma^i_{kp} x^p_\beta n^p$$

where $n^i$ are the components of the normal to $S$ in $M^4$ and $b_{\alpha\beta}$ is the covariant tensor of the second fundamental form. Using Eq. 2.147 in Eq. 2.146 and taking the inner multiplication of every term with $a^{\alpha\beta}$, we get

$$a^{\alpha\beta} x^i_{\alpha;\beta} + (\Delta_2 g^\gamma_{\gamma}) x^i_{,\gamma} = -g^{rk} \Gamma^i_{rk} + P n$$

(2.148)

where

$$\Delta_2 = a^{\alpha\beta} \left( \frac{\partial^2}{\partial x^\alpha \partial x^\beta} - \Upsilon^\gamma_{\alpha\beta} \frac{\partial}{\partial x^\gamma} \right)$$

and

$$P = a^{\alpha\beta} b_{\alpha\beta}$$

Eq. 2.148 is a generalization of Eq. 2.132 for a Riemannian hypersurface [32, 33]. The main difference is the appearance of the space Christoffel symbols, which vanish when $M^4$ becomes $E^3$. 

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2.6 Concluding Remarks

1. If Dirichlet data is prescribed on the bounding curves of a given surface, then the three scalar equations from Eq. 2.132 can be used to generate coordinates in the surface. The distribution of these coordinates can be controlled by assigning suitable functions $P$ and $Q$.

2. If the coefficients of the first and the second fundamental forms have been given as functions of some surface coordinates, then the surface suitable to these coefficients can be generated by solving the three scalar equations from Eq. 2.132. In this case, $\Delta_\xi u^\xi$ is expressed in terms of the given $g_{\alpha\beta}$ and $k_1 + k_{\bar{1}}$ is expressed in terms of $g^{\alpha\beta} P_{\alpha\beta}$.

3. For a recent account of the use of elliptic equations in grid generation with algebraic parametric transformations, refer to [34].

References